South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 3 (2025), pp. 81-98

DOI: 10.56827/SEAJMMS.2025.2103.6 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

MODIFIED TYPE-1 DIRICHLET AVERAGES OF THE THREE-PARAMETER MITTAG-LEFFLER FUNCTION THROUGH FRACTIONAL INTEGRALS AND SPECIAL FUNCTIONS

Princy T. and Nicy Sebastian*

Department of Statistics,
Cochin University of Science and Technology,
Cochin - 682022, Kerala, INDIA

E-mail : princyt.t@gmail.com

*Department of Statistics,
St. Thomas College (Autonomous)
Thrissur - 680001, Kerala, INDIA

E-mail : nicycms@gmail.com

(**Received:** *May 09, 2025* **Accepted:** *Dec. 23, 2025* **Published:** *Dec. 30, 2025)*

Abstract: The classical power means of Hardy, Littlewood and Polya, which contains the harmonic mean, arithmetic mean and geometric mean, is generalized to the Y-mean and hypergeometric mean by Carlson. Carlson's hypergeometric mean is to average a function over a type-1 Dirichlet measure, and this term in the current literature is known as the Dirichlet average of that function. The present paper introduces a new Dirichlet average, associated with the modified type-1 Dirichlet measures called modified type-1 Dirichlet averages. This paper also investigates the modified type-1 Dirichlet averages of a three-parameter Mittag-Leffler type function, which is expressed using Riemann-Liouville integrals and hypergeometric functions with multiple variables.

Keywords and Phrases: Dirichlet average, generalized type-1 and type-2 Dirichlet models, Mittag-Leffler functions, Riemann-Liouville fractional integrals, hypergeometric functions of one and many variables.

2020 Mathematics Subject Classification: 62E15, 60E05, 33E12, 26A33, 33C70, 33C20, 33C65.

1. Introduction

The classical class of power means, see Hardy et al. (1952), contains the harmonic, arithmetic and geometric means. Let $w' = (w_1, ..., w_n), z' = (z_1, ..., z_n)$ where a prime denotes the transpose, and let a be a scalar quantity. Let $z_j > 0, w_j > 0, j = 1, ..., n$ and $\sum_{j=1}^{n} w_j = 1$. Then, the classical weighted average is

$$z_0(a, w) = [w_1 z_1^a + ... + w_n z_n^a]^{\frac{1}{a}}.$$

For $a = 1$, z_0 yields $\sum_{j=1}^{n} w_j z_j$ or the arithmetic mean; when $a = -1$ it gives $[\sum_j (\frac{w_j}{z_j})]^{-1}$ = the harmonic mean and when $a \to 0_+$ then it yields $\prod_{j=1}^{n} z_j^{w_j}$ = the geometric mean. This weighted mean z_0 is generalized to Y-mean by deFinetti, see deFinetti (1974) and to hypergeometric mean by Carlson, see Carlson (1969, 1977). Carlson's generalization involves assuming a type-1 Dirichlet measure for the weights $(w_1, ..., w_{n-1})$, and then the average of a given function is taken over this Dirichlet measure. In the current literature, this is known as the Dirichlet average of that function.

The Mittag-Leffler function is an essential special function in mathematics, particularly in the study of fractional calculus and various applied sciences. It generalizes the exponential function and serves as a powerful tool in describing complex systems with non-exponential growth or decay. It has been referred to as the queen function in fractional calculus by Mainardi and Gorenflo, see Mainardi and Gorenflo(2007) and Mainardi (2020). The three parameter Mittag-Leffler function is given by

$$E_{\alpha,\delta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \delta) k!}, \tag{1}$$

where $\alpha, \delta, \gamma \in \mathbb{C}, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}, \Re(\alpha) > 0, \Re(\delta) > 0$ and $(\gamma)_k$ is the Pochhammer symbol

$$(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma + 1) \dots (\gamma + k - 1), k \in \mathbb{N}, \gamma \neq 0. \tag{2}$$

The generalised Mittag-Leffler function was first introduced by Prabhakar in 1971, with its properties later studied by Prabhakar (1971) and Kilbas et al. (2002, 2004).

The study of Dirichlet averages of three parameter Mittag-Leffler type functions is important in both theoretical mathematics and its applications in various fields. Mittag-Leffler functions generalize the exponential function, which is crucial in describing a wide range of phenomena such as relaxation processes in physics, diffusion processes, and anomalous dynamics. Studying their Dirichlet averages provides deeper insight into generalized functions that extend beyond simple exponential behaviour. The three parameter Mittag-Leffler function, combined with Dirichlet averages, can lead to new solutions for fractional differential equations applicable in physics, control theory, and engineering. Special functions like the Mittag-Leffler function arise naturally in many physical problems, including quantum mechanics, fluid dynamics, and thermodynamics. The Dirichlet averages of these functions offer a more refined comprehension of their behaviour across many domains and contexts. For more details see Kilbas et al. (2004), Kilbas and Kattuveettill (2008), Rogosin (2015) and Kumar et al. (2022). In stochastic processes, generalized Mittag-Leffler functions are often used to describe waiting times in renewal processes and other random phenomena. Understanding their Dirichlet averages could be useful in improving models of random events in fields such as finance and network theory, see Stamova and Stamov (2017). In mathematical physics, Dirichlet averages of Mittag-Leffler functions can describe relaxation phenomena in complex systems, including anomalous diffusion in disordered media, viscoelastic materials, and biological systems. Such studies help in modelling real-world physical processes more accurately; see Mainardi (2010), Cahoy and Polito (2013), Gorenflo et al. (2014). The study of three parameter Mittag-Leffler functions through Dirichlet averages can be extended to higher-dimensional problems in geometry and topology, which are important for understanding complex dynamical systems and multidimensional physical systems; see Agarwal et al. (2024).

The paper presents the Dirichlet averages of various functions in various generalized Dirichlet measures. This paper is structured as follows: Section 2 explores Dirichlet averages of various functions over different types of Dirichlet measures and introduces a new modified type-1 Dirichlet measure. In Sections 3 and 4, we present a modified type-1 Dirichlet average of the three parameter Mittag-Leffler function using fractional integrals, expressed in terms of the Srivastava–Daoust function. Finally, Section 5 offers concluding remarks and suggestions for future work.

2. Dirichlet averages

This section examines Dirichlet averages of different functions across various types of Dirichlet measures. Consider a real type-1 Dirichlet density

$$f_1(x_1, ..., x_k) = D_k x_1^{\alpha_1 - 1} ... x_k^{\alpha_k - 1}(1 - x_1 - ... - x_k)^{\alpha_{k+1} - 1}, \tag{3}$$

for $\alpha_j > 0, j = 1, ..., k+1, (x_1, ..., x_k) \in \Omega, \Omega = \{(x_1, ..., x_k) | 0 \le x_j \le 1, j = 1, ..., k, 0 \le x_1 + ... + x_k \le 1\}$ and $f_1(x_1, ..., x_k) = 0$ elsewhere, where

$$D_k = \frac{\Gamma(\alpha_1 + ... + \alpha_{k+1})}{\Gamma(\alpha_1)...\Gamma(\alpha_{k+1})}. \tag{4}$$

and $\Gamma(\cdot)$ denotes the gamma function. Let $\phi_1(x_1, ..., x_k)$ be a function of $x_1, ..., x_k$ such that the integral

$$A_1 = \int_{\Omega} \phi_1(x_1, ..., x_k) f_1(x_1, ..., x_k) \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_k, \tag{5}$$

exists. Here, the function ϕ_1 is averaged over the Dirichlet measure in (3) and hence A_1 is called the Dirichlet average of ϕ_1. In statistical language, A_1 is the expected value of ϕ_1 or $A_1 = E(\phi_1)$ where $E(\cdot)$ denotes the expected value of $(\cdot)$. We may also consider averaging ϕ_1 in a real type-2 Dirichlet measure. The real type-2 Dirichlet measure in the real scalar variables case is the following:

$$f_2(x_1, ..., x_k) = D_k x_1^{\alpha_1 - 1} ... x_k^{\alpha_k - 1} (1 + x_1 + ... + x_k)^{-(\alpha_1 + ... + \alpha_k + \alpha_{k+1})}, \tag{6}$$

where $\Re(\alpha_j) > 0, j = 1, ..., k+1, x_j \geq 0, j = 1, ..., k,$ D_k is the same normalizing constant appearing in (4), and zero elsewhere. Let $\phi_2 = \phi_2(x_1, ..., x_k)$ be a function of $x_1, ..., x_k$ then the Dirichlet average of ϕ_2 over the Dirichlet measure in (5) is given by the following as an expected value:

$$A_2 = E[\phi_2] = \int_{x_1=0}^{\infty} ... \int_{x_k=0}^{\infty} \phi_2 f_2(x_1, ..., x_k) \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_k, \tag{7}$$

provided this expected value exists.

In the literature, Dirichlet average usually refers to averaging over the type-1 Dirichlet density in (3). We will consider type-1, type-2 and their generalised forms as the Dirichlet measure and consider averaging various functions over these generalised Dirichlet measures. There are several generalisations of type-1 and type-2 Dirichlet measures, given by Mathai and his co-researchers; see, for example, Mathai (2003), Thomas and Jacob (2006), Thomas and Mathai (2009). One such generalisation is the following:

$$\begin{aligned} f_3(x_1, ..., x_k) &= C_k x_1^{\alpha_1 - 1} (1 - x_1)^{\beta_1} x_2^{\alpha_2 - 1} (1 - x_1 - x_2)^{\beta_2} ... \\ &\times x_k^{\alpha_k - 1} (1 - x_1 - ... - x_k)^{\beta_k + \alpha_{k+1} - 1}, \end{aligned} \tag{8}$$

with $(x_1, ..., x_k) \in \Omega$ of (8) and $f_3(x_1, ..., x_k) = 0$ elsewhere, where C_k is the normalizing constant. While working on order statistics from generalized logistic models, Mathai ended up with the model in (8). Later, it was realized that this model was already derived by Connor and Mosimann (1969) by using the neutrality principle of proportions. A lot of recent works in this direction, generalizing the neutrality principle and characterizing Dirichlet density by using such generalized concepts, etc., are there; for example, Ongaro and Migliorati (2013). Note that when

$\beta_1 = 0, ..., \beta_k = 0$ in (8), we have the Dirichlet measure in (3). Averaging a function $\phi_3(x_1, ..., x_k)$ over (8) is a generalized Dirichlet average. Let us consider a few examples of averaging functions over the Dirichlet measure (3) first. Let $\phi_1 = \phi_1(x_1, x_2) = \mathrm{e}^{-ax_1}, a > 0$. Let us average it over (3) for $k = 2$. Then

$$\begin{aligned} E[\phi_1] &= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_\Omega \mathrm{e}^{-ax_1} x_1^{\alpha_1-1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &= \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \frac{\Gamma(\alpha_1+k)}{\Gamma(\alpha_1)} \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1+k+\alpha_2+\alpha_3)}. \end{aligned}$$

where the integrals over x_1 and x_2 are evaluated by using a type-1 Dirichlet integral, or available from the normalizing constant in (4). But $\Gamma(\alpha_1 + k) = \Gamma(\alpha_1)(\alpha_1)_k$ and $\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + k) = \Gamma(\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)_k$. Then

$$E[\phi_1] = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \frac{(\alpha_1)_k}{(\alpha_1+\alpha_2+\alpha_3)_k} = {}_1F_1(\alpha_1; \alpha_1+\alpha_2+\alpha_3; -a),$$

where ${}_1F_1$ is a confluent hypergeometric series.

Consider averaging $\phi_2 = \mathrm{e}^{-a_1x_1-a_2x_2}$, $a_1 > 0, a_2 > 0$, over the Dirichlet measure in (3) for $k = 2$. Then proceeding as above we have

$$\begin{aligned} E[\phi_2] &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(-a_1)^{k_1}}{k_1!} \frac{(-a_2)^{k_2}}{k_2!} \frac{(\alpha_1)_{k_1}(\alpha_2)_{k_2}}{(\alpha_1+\alpha_2+\alpha_3)_{k_1+k_2}} \\ &= F_B(\alpha_1, \alpha_2; \alpha_1+\alpha_2+\alpha_3; -a_1, -a_2), \end{aligned}$$

where F_B is Appell's function F_B, see also Mathai (1993). Consider a general case. Let $\phi_3(x_1, ..., x_k) = \mathrm{e}^{-a_1x_1-...-a_kx_k}$. Then proceeding as above we have the following result:

$$\begin{aligned} E[\phi_3] &= \sum_{m_1=0}^{\infty} ... \sum_{m_k=0}^{\infty} \frac{(-a_1)^{m_1}}{m_1!} ... \frac{(-a_k)^{m_k}}{m_k!} \frac{(\alpha_1)_{m_1}...(\alpha_k)_{m_k}}{(\alpha_1+...+\alpha_{k+1})_{m_1+...+m_k}} \\ &= f_B(\alpha_1, ..., \alpha_k; \alpha_1+...+\alpha_{k+1}; -a_1, ..., -a_k), \end{aligned}$$

where f_B is Lauricella function f_B, see for example, Mathai (1993).

2.1. Modified type-1 Dirichlet measure and Dirichlet averages

Let us make the transformation $x_j = y_j^2, -1 \le y_j \le 1, 0 \le x_j \le 1, j = 1, ..., k$ in (3) such that $0 < y_1^2 + ... + y_k^2 \le 1$. Then the model in (3) becomes the following:

$$\begin{aligned} g_1(y_1, ..., y_k)\mathrm{d}y_1 \wedge ... \wedge \mathrm{d}y_k &= \frac{\Gamma(\alpha_1+...+\alpha_{k+1})}{\Gamma(\alpha_1)...\Gamma(\alpha_{k+1})}|y_1|^{2\alpha_1-1}...|y_k|^{2\alpha_k-1} \\ &\times [1-y_1^2-...-y_k^2]^{\alpha_{k+1}-1}\mathrm{d}y_1 \wedge ... \wedge \mathrm{d}y_k, \quad (9) \end{aligned}$$

for $-1 \leq y_j \leq 1, j = 1, ..., k, 0 \leq y_1^2 + ... + y_k^2 \leq 1$. Consider the special case $\alpha_1 = \frac{1}{2} = ... = \alpha_k$. Then the model in (9) is the following:

$$g_2(y_1, ..., y_k) = \frac{\Gamma(\frac{k}{2} + \alpha_{k+1})}{\pi^{\frac{k}{2}}\Gamma(\alpha_{k+1})}[1 - y_1^2 - ... - y_k^2]^{\alpha_{k+1}-1}. \tag{10}$$

For convenience let us write $\alpha_{k+1} = \beta > 0$. This modified type-1 Dirichlet measure has many interesting properties and connections to problems in various areas.

3. Modified type-1 Dirichlet average of three parameter Mittag-Leffler function using fractional integrals

In this section, we study the generalized Dirichlet average of the three parameter Mittag-Leffler function (1) in the forms

$$M_{\alpha,\delta}^{\gamma}(\beta, \beta'; x, y) = \int_{E_1} E_{\alpha,\delta}^{\gamma}(u \circ z) \mathrm{d}\mu_{\beta,\beta'}(u), \tag{11}$$

for $\alpha, \delta, \gamma, \beta, \beta' \in \mathbb{C}, \Re(\alpha) > 0; \Re(\beta) > 0, \Re(\beta') > 0; x, y \in \mathbb{R}$, and

$$M_{\beta,\delta}^{\gamma,\alpha}(b_1, \ldots, b_n; 1 - z_1, \ldots, 1 - z_n) = \int_{E_{n-1}} E_{\beta,\delta}^{\gamma}((1 - u \circ z)^{\alpha}) \mathrm{d}\mu_b(u), \tag{12}$$

and $\alpha, \beta, \delta, \gamma \in \mathbb{C}, \Re(\beta) > 0; b_i, z_i \in \mathbb{C}, \Re(b_i) > 0, i = 1, 2, \ldots, n$. Here E_{n-1} is the simplex in $\mathbb{R}^{n-1}, n \geq 2$:

$$E_{n-1} = \{(u_1, u_2, \ldots, u_n) : -1 \leq u_i \leq 1, u_1^2 + u_2^2 + \cdots + u_{n-1}^2 \leq 1\}, \tag{13}$$

$$u \circ z = \sum_{i=1}^{n-1} u_i^2 z_i + (1 - u_1^2 - \cdots - u_{n-1}^2) z_n, \tag{14}$$

and $\mathrm{d}\mu_b$ is defined by

$$\mathrm{d}\mu_b(u) = \frac{1}{B(b)} |u_1|^{2b_1 - 1} \ldots |u_{n-1}|^{2b_{n-1}-1} [1 - u_1^2 - \cdots - u_{n-1}^2]^{b_n - 1} \mathrm{d}u_1 \cdots, \mathrm{d}u_{n-1}, \tag{15}$$

where

$$B(b) = \frac{\Gamma(b_1) \cdots \Gamma(b_n)}{\Gamma(b_1 + \cdots + b_n)}, (\Re(b_j) > 0, j = 1, \ldots, n,). \tag{16}$$

The general Dirichlet average of a function $f(z) = f(z_1, \cdots, z_n)$ was defined by Carlson (1969) in the form

$$F(b, z) = \int_{E_{n-1}} f(u \circ z) \mathrm{d}\mu_b(u), \tag{17}$$

where $\mathrm{d}\mu_b(u)$ is defined in (15). For $n = 1, F(b; z) = f(z)$. Carlson (1969) investigated the average (17) for $f(z) = z^k$ with any real $k \in \mathbb{R}$ in the form

$$R_k(b, z) = \int_{E_{n-1}} (u \circ z)^k \mathrm{d}\mu_b(u). \tag{18}$$

In particular, if $n = 2$, we note that we have a symmetric form of Carlson's (1969) [3] Dirichlet average obtained via the transformation $x = u^2$, $u \in [-1, 1]$. Consequently, the associated normalizing constants and integral representations differ from Carlson's original simplex-based formulation by a fixed multiplicative factor, which is absorbed into our definition of the modified type-1 Dirichlet average.

$$R_k(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_{-1}^{1} \left[u^2 x + (1 - u^2)y\right]^k |u|^{2\beta-1}(1 - u^2)^{\beta'-1}\mathrm{d}u, \tag{19}$$

where $\beta, \beta' \in \mathbb{C}$ are complex numbers with positive real parts with $\Re(\beta) > 0, \Re(\beta') > 0$ and $x, y \in \mathbb{R}$.

We prove the representation for (11) and (12) in terms of the Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, Samko et al. (1993), Kilbas et al. (2006):

$$(I^{\alpha}_{0+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)dt, \quad x > a, a \in \mathbb{R}. \tag{20}$$

3.1. Representation of R_k and $M^{\gamma}_{\alpha,\beta}$ in terms of fractional integrals

In this section, we deduce representations for the Generalized Dirichlet averages $R_k(\beta, \beta'; x, y)$ and $M^{\gamma}_{\alpha,\delta}(\beta, \beta'; x, y)$ via the fractional integral (20). Such a representation for the former is produced by the first assertion.

Proposition 3.1. *Let x, y be such that $y > x \geq 0, \beta, \beta' > 0$ and $k \in \mathbb{R}$. Let R_k and I^{α}_{0+} be given by (19) and (20), respectively. Then there hold the following formula,*

$$R_k(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)})(y - x)^{(1-\beta-\beta')} \left(I^{\beta'}_{0+}(t^{\beta-1}(y - t)^k)\right)(y - x). \tag{21}$$

Proof. For $y > x > 0$ rewrite $R_k(\beta, \beta'; x, y)$ given by (19) in the form

$$R_k(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_{-1}^{1} \left[u^2 x + (1 - u^2)y\right]^k (u)^{2\beta-1}(1 - u^2)^{\beta'-1}\mathrm{d}u. \tag{22}$$

Changing the variable $u^2 = v$, we have

$$R_k(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \left[vx + (1 - v)y\right]^k v^{\beta-1}(1 - v)^{\beta'-1}\mathrm{d}v. \tag{23}$$

Making the change of variables $v(y-x)=t$, we have

$$R_k(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}(y-x)^{(1-\beta-\beta')}\int_0^{y-x}(y-t)^k t^{\beta-1}(y-x-t)^{\beta'-1}\mathrm{d}t. \quad (24)$$

According to (20), this proves (21).
Let ${}_2F_1(a,b;c;z)$ be the Gauss hyper geometric function defined for complex parameters $a,b,c\in\mathcal{C}(c\neq 0,-1,\dots)$ by the hypergeometric series

$$ {}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad (25)$$

which converges absolutely for $|z|<1$ and $|z|=1,\Re(c-a-b)>0$, for details, see Erdélyi et al. (1981). The next result gives the representation of $R_k(\beta,\beta';x,y)$ in terms of (25).

Proposition 3.2. *Let the conditions of Proposition 1 hold and let* $|1-\frac{x}{y}|<1$. *Then there holds the following representation*

$$R_k(\beta,\beta';x,y)=y^k{}_2F_1(-k,\beta;\beta+\beta';1-\frac{x}{y}). \quad (26)$$

Proof. Taking out y^k from the integrand in (23), we have

$$\begin{aligned}R_k(\beta,\beta';x,y) &= y^k\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\\ &\times \int_0^1\left[1-v(1-\frac{x}{y})\right]^k v^{\beta-1}(1-v)^{\beta'-1}\mathrm{d}v. \quad (27)\end{aligned}$$

According to the well known representation of Gauss hypergeometric function

$$ {}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\mathrm{d}x, \quad (28)$$

for $0<\Re(b)<\Re(c);|\arg(1-z)|<\pi$. By using this integral we can easily establish (27).

Theorem 3.1. *Let* $\alpha,\delta,\gamma,\beta,\beta'\in\mathcal{C}$ *be complex numbers,* $\Re(\alpha)>0,\Re(\beta)>0,\Re(\beta')>0$ *and let* $x,y\in\mathbb{R}$ *be real numbers such that* $y>x\geq 0$, *and let* $M^{\gamma}_{\alpha,\delta}$ *and* $I^{\beta'}_{0+}$ *be given by (11) and (19) respectively. Then the Dirichlet average of Mittag-Leffler function is given by*

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}\left[I^{\beta'}_{0+}(t^{\beta-1}E^{\gamma}_{\alpha,\delta}(y-t))\right](y-x). \quad (29)$$

Proof. According to (11) and (1) we have

$$\begin{aligned} M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_{-1}^{1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha k+\delta)} \\ &\times [u^2x+(1-u^2)y]^k|u|^{2\beta-1}(1-u^2)^{\beta'-1}\mathrm{d}u \text{ put } u^2=v, \end{aligned}$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_{0}^{1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha k+\delta)}[y+v(x-y)]^k v^{\beta-1}(1-v)^{\beta'-1}\mathrm{d}v. \quad (30)$$

Changing the orders of integration and summation, we get

$$\begin{aligned} M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha k+\delta)} \\ &\times \int_{0}^{1}[y+v(x-y)]^k v^{\beta-1}(1-v)^{\beta'-1}\mathrm{d}v. \end{aligned}$$

Changing the variable $v(y-x)=t$, and taking (19) into account, we have

$$\begin{aligned} M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha k+\delta)}(y-x)^{1-\beta-\beta'} \\ &\times \int_{0}^{y-x}[y-t)]^k t^{\beta-1}(y-x-t)^{\beta'-1}\mathrm{d}t \end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}(y-x)^{1-\beta-\beta'}\int_{0}^{y-x}E^{\gamma}_{\alpha,\delta}(y-t)t^{\beta-1}(y-x-t)^{\beta'-1}\mathrm{d}t \\ &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}\left[I^{\beta'}_{0+}(t^{\beta-1}E^{\gamma}_{\alpha,\delta}(y-t))\right](y-x). \end{aligned}$$

This completes the proof.

3.2. Special cases

In this section we discuss some particular cases of Theorem 3.1.

Corollary 3.1. *Let the conditions of Theorem* 3.1 *be satisfied and* $\alpha=1$*, Then*

$$M^{\gamma}_{1,\delta}(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}\left[I^{\beta'}_{0+}(t^{\beta-1}{}_1F_1(\gamma;\delta;(y-t)))\right](y-x). \quad (31)$$

Let $\gamma=1$, in Theorem 3.1, we get the next result

Corollary 3.2. *Let the conditions of Theorem* 3.1 *be satisfied and* $\gamma = 1$. *Then*

$$M^1_{\alpha,\delta}(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'}\left[I^{\beta'}_{0+}(t^{\beta-1}E_{\alpha,\delta}(y-t))\right](y-x). \quad (32)$$

Corollary 3.3. *Let the conditions of Theorem* 3.1 *be true with* $\alpha > 0$ *and* $y = 0$. *Then*

$$M^1_{\alpha,\delta}(\beta,\beta';x,0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)}{}_2\Psi_2\left[\begin{array}{cc} (\gamma,1), & (\beta,1) \\ (\beta+\beta',1), & (\delta,\alpha) \end{array} \mid x\right], x > 0. \quad (33)$$

$$M^1_{\alpha,\delta}(\beta,\gamma-\beta;x,0) = \frac{1}{\Gamma(\beta)}{}_1\Psi_1\left[\begin{array}{c} (\beta,1) \\ (\delta,\alpha) \end{array} \mid x\right], x > 0. \quad (34)$$

where ${}_p\psi_q$ *is the generalized Wright hypergeometric function.*
The generalized Wright hypergeometric function is defined for complex $a_i, b_j \in \mathcal{C}$ and real $A_i, B_j \in \mathbb{R}, i = 1,2\ldots p; j = 1,2,\ldots q$ by the series

$${}_p\Psi_q\left[\begin{array}{ccc} (b_1,B_1), & \ldots, & (b_q,B_q) \\ (a_1,A_1), & \ldots, & (a_p,A_p) \mid z \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_in)z^n}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)n!}. \quad (35)$$

This function was first introduced by Wright (1935).

Proof 3.1. Using (30), changing the orders of integration and summation and applying the formulas of connections between the Pochhammer symbol and the beta function with the gamma function

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}, (z \in \mathcal{C}, k \in \mathbb{N}_0), B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, (\alpha,\beta \in \mathcal{C}), \quad (36)$$

we have

$$\begin{aligned} M^\gamma_{\alpha,\delta}(\beta,\beta';x,0) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_nx^n}{\Gamma(\alpha n+\delta)n!}\int_0^1 v^{n+\beta-1}(1-v)^{\beta'-1}\mathrm{d}v \\ &= \frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_nx^n}{\Gamma(\alpha n+\delta)n!}\frac{\Gamma(n+\beta)\Gamma(\beta')}{\Gamma(n+\beta+\beta')} \\ &= \frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)x^n}{\Gamma(\alpha n+\delta)n!}\frac{\Gamma(n+\delta)}{\Gamma(n+\beta+\beta')}, \end{aligned}$$

whenever the gammas are defined. By using the definition of Generalized Wright hypergeometric function we can easily establish this result.

Corollary 3.4. *Let the conditions of Theorem* 3.1 *be hold with* $\alpha > 0$ *and* $x = 0$. *Then*

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';0,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)} {}_2\Psi_2\left[\begin{array}{cc} (\gamma,1), & (\beta',1) \\ (\beta+\beta',1), & (\delta,\alpha) \end{array} \mid -y\right]. \tag{37}$$

In particular, when $\beta + \beta' = \gamma$,

$$M^{\gamma}_{\alpha,\delta}(\beta,\gamma-\beta;0,y) = {}_1\Psi_1\left[\begin{array}{c} (\gamma-\beta,1) \\ (\delta,\alpha) \end{array} \mid -y\right]. \tag{38}$$

Corollary 3.5. *Let the conditions of Corollaries* 3.3 *and* 3.4 *be valid and let* $\alpha = 1$. *Then*

$$M^{\gamma}_{1,\delta}(\beta,\beta';x,0) = \frac{1}{\Gamma(\delta)} {}_2F_2(\gamma,\beta;\beta+\beta',\delta;x) \quad x > 0, \tag{39}$$

$$M^{\gamma}_{1,\delta}(\beta,\beta';0,y) = \frac{1}{\Gamma(\delta)} {}_2F_2(\gamma,\beta';\beta+\beta',\delta;y) \quad y < 0, \tag{40}$$

where ${}_2F_2$ is the special case of generalized hypergeometric function ${}_pF_q(z)$. It is defined for $a_1,\ldots,a_p,b_1,\ldots,b_q \in \mathcal{C}(b_j \neq 0,-1,-2,\ldots; j = 1,2,\ldots,q)$ by the generalized hypergeometric function series[Erdélyi et al. (1981)]

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k,\ldots(a_p)_k}{(b_1)_k,\ldots(b_q)_k}\frac{z^k}{k!}. \tag{41}$$

The series is absolutely convergent for any $z \in \mathcal{C}$ when $p \leq q$. Setting $\alpha = \gamma = 1, \beta' = 1-\beta$, from (39) and (40) we have the following result.

Corollary 3.6. *Let the conditions of Corollary* 3.5 *be satisfied and for* $\alpha = 1$ *and* $\beta' = 1-\beta$. *Then*

$$M^{1}_{1,\delta}(\beta,1-\beta;x,0) = \frac{1}{\Gamma(\delta)} {}_1F_1(\beta;\delta;x) \quad x > 0. \tag{42}$$

$$M^{1}_{1,\delta}(\beta,1-\beta;0,y) = \frac{1}{\Gamma(\delta)} {}_1F_1(1\beta;\delta;y) \quad y < 0. \tag{43}$$

4. Modification of $M^{\gamma,\alpha}_{\alpha,\delta}(\beta,\beta';x,y)$ in terms of hypergeometric function

A modified Dirichlet average is defined as from (1) and is presented in this section as follows:

$${}_\rho M^{\gamma;\alpha}_{\alpha,\delta}(\beta,\beta';x,y) = \int_{E_2} (u \circ z)^{\rho-1} E^{\gamma}_{\alpha,\beta}[(u \circ z)^{\alpha}] \mathrm{d}\mu_{\beta,\beta'}(u). \tag{44}$$

The following theorem is an analogue of Theorem 3.1, providing representation for (44) holds.

Theorem 4.1. *Let $\alpha, \delta, \gamma, \rho, \beta, \beta' \in \mathcal{C}$ be complex numbers, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\beta') > 0$ and let $x, y \in \mathbb{R}$ be real numbers such that $y > x$. Then there holds the following relation:*

$$ {}_\rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'} \left[I^{\beta'}_{y+}(t^{\rho-1}(t-y)^{\beta-1}E^{\gamma}_{\alpha,\delta}(t^{\alpha})) \right](x). \quad (45) $$

Proof. By combining (44) and (19) and switching the integration and summation orders, we have

$$ \begin{aligned} {}_\rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} \int_{-1}^{1} [u^2x + (1-u^2)y]^{\rho-1} E^{\gamma}_{\alpha,\delta}([u^2x + (1-u^2)y]^{\alpha}) \\ &\times \; |u|^{2\beta-1}(1-u^2)^{\beta'-1}\mathrm{d}u \end{aligned} $$

$$ = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k+\delta)} \int_0^1 [y + v(x-y)]^{\alpha k+\rho-1} v^{\beta-1}(1-v)^{\beta'-1}\mathrm{d}v $$

Let $y + v(x-y) = t$ and using the definition of Mittag-Leffler function, we get

$$ \begin{aligned} {}_\rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'} \\ &\times \int_y^x t^{\rho-1}(t-y)^{\beta-1}E^{\gamma}_{\alpha,\delta}(t^{\alpha})(x-t)^{\beta'-1}\mathrm{d}t. \end{aligned} $$

This results in the outcome in (44) according to (20) when $n = 2$.

Corollary 4.1. *Let the conditions of Theorem 4.1 be valid with $\gamma = 1$, then*

$$ \begin{aligned} {}_\rho M^{1,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(y-x)^{1-\beta-\beta'} \\ &\times \left[I^{\beta'}_{y+}(t^{\rho-1}(t-y)^{\beta-1}E^{1}_{\alpha,\delta}(t^{\alpha})) \right](x), (y > x). \end{aligned} \quad (46) $$

Setting $\delta = \beta + \rho - 1$ and $y = 0$ in Corollary 4.1, we get the following result.

Corollary 4.2. *Let the conditions of Corollary 4.1 be satisfied , $\delta = \beta + \rho - 1$, and let $x > 0$ and $y = 0$. Then*

$$ {}_\rho M^{1,\alpha}_{\alpha,\beta+\rho-1}(\beta, \beta'; x, 0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} x^{\rho-1} E_{\alpha,\beta+\rho+\beta'-1}(x^{\alpha}), (x > 0). \quad (47) $$

In particular if $\beta' = 1-\beta$, then

$$ {}_\rho M^{1;\alpha}_{\alpha,\delta}(\beta, 1-\beta; x, 0) = \frac{1}{\Gamma(\beta)} x^{\rho-1} E_{\alpha,\rho}(x^\alpha), (x > 0). \tag{48} $$

Proof. Taking $\delta = \beta + \rho - 1$ and $y = 0$ in (46), using (20) and changing the order of integration and summation, we have

$$ {}_\rho M^{1,\alpha}_{\alpha,\beta+\rho-1}(\beta, \beta'; x, 0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} x^{1-\beta-\beta'} \left[I^{\beta'}_{y+}(t^{\beta+\rho-1}(t-y)^{\beta-1} E_{\alpha,\beta+\rho-1}(t^\alpha)) \right](x) $$

$$ = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} x^{1-\beta-\beta'} \int_0^x (x-t)^{\beta'-1} t^{\beta+\rho-2} \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{k!\Gamma(\alpha k + \beta + \rho - 1)} \mathrm{d}t $$

$$ = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} x^{-\beta} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\alpha k + \beta + \rho - 1)} \int_0^x (1 - \frac{t}{x})^{\beta'-1} t^{\alpha k + \beta + \rho - 2} \mathrm{d}t. $$

put $t = ux$ and using the definition of beta integral, we have

$$ {}_\rho M^{1,\alpha}_{\alpha,\beta+\rho-1}(\beta, \beta'; x, 0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} x^{-\beta} \sum_{k=0}^{\infty} \frac{x^{\alpha k + \beta + \rho - 1}}{k!\Gamma(\alpha k + \beta + \rho - 1)}. $$

Now, we can easily prove (47) by using the definition of Mittag-Leffler function.

Corollary 4.3. *Let the conditions of Corollary* 4.1 *be satisfied, $\rho = 1, \delta = \beta$, and let $x > 0$ and $y = 0$. Then*

$$ {}_1 M^{1,\alpha}_{\alpha,\beta}(\beta, \beta'; x, 0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} E_{\alpha,\beta+\beta'}(x^\alpha) \quad (x > 0). \tag{49} $$

In particular, if $\beta' = 1-\beta$, then

$$ {}_1 M^{1,\alpha}_{\alpha,\beta}(\beta, 1-\beta; x, 0) = \frac{1}{\Gamma(\beta)} E_\alpha(x^\alpha) \quad (x > 0), \tag{50} $$

where

$$ E_\alpha(z) = E_{\alpha,1}(z). \tag{51} $$

The proof is similar to the proof of Corollary 4.2.

In conclusion of this section, we examine a modification of (11) in the form

$$ \rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) = \int_E (u \circ z)^{\rho-1} E^\gamma_{\alpha,\beta}[(u \circ z)^\alpha] \, d\mu_\delta(u). \tag{52} $$

The subsequent sentence provides the expression for (52) with $\rho = \delta$.

Theorem 4.2. *Let $\alpha, \delta, \gamma, \beta, \beta' \in \mathcal{C}$ be complex numbers, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\beta') > 0$, and let $x, y \in \mathbb{R}, y > x$ be real numbers. Then there holds the following relation:*

$$ {}_{\delta}M^{\gamma;\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) = \sum_{k=0}^{\infty} \frac{(\gamma)_k y^{\alpha k+\delta-1}}{k!\Gamma(\alpha k + \delta)} {}_2F_1(\beta, -\alpha k - \delta + 1; \beta + \beta'; 1 - \frac{x}{y}). \tag{53} $$

Proof. By utilising (52), altering the orders of integration and summation, and employing the integral representation (27), we obtain

$$ {}_{\delta}M^{\gamma;\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) $$

$$ = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k + \delta)} [y + v(x - y)]^{\alpha k+\delta-1} v^{\beta-1}(1 - v)^{\beta'-1} \mathrm{d}v $$

$$ = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{k=0}^{\infty} \frac{(\gamma)_k y^{\alpha k+\delta-1}}{k!\Gamma(\alpha k + \delta)} \int_0^1 [1 - v(1 - \frac{x}{y})]^{\alpha k+\delta-1} v^{\beta-1}(1 - v)^{\beta'-1} \mathrm{d}v $$

$$ = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{k=0}^{\infty} \frac{(\gamma)_k y^{\alpha k+\delta-1}}{k!\Gamma(\alpha k + \delta)} \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\beta + \beta')} {}_2F_1(\beta, -\alpha k - \delta + 1; \beta + \beta'; 1 - \frac{x}{y}). $$

5. Conclusion and future work

Several eminent mathematicians have provided the Dirichlet average for simple functions such as power and exponential functions, among others. We have taken the modified type-1 Dirichlet average of those functions, applied a fractional integral, and obtained new results after converting the basic function into the summation form. Thus, a link between the modified type-1 Dirichlet average of a function and the fractional integral has been discovered. The three-parameter Mittag-Leffler-type function's modified type-1 Dirichlet average was explored. In Theorems 3.1 and 4.1, the bivariate Dirichlet averages of the three-parameter Mittag-Leffler-type functions were expressed in terms of the Riemann-Liouville fractional integral. In Theorem 4.2, the bivariate Dirichlet average of the three-parameter Mittag-Leffler type function was shown to be expressed in terms of the Srivastava-Daoust generalisation of the Lauricella hypergeometric function. The conclusions of the theorems demonstrate the capacity to yield several unique examples owing to the general characteristics of the three-parameter Mittag-Leffler function. It is expected that scientists and mathematicians will make use of these findings. The multivariate Dirichlet average of the three-parameter Mittag-Leffler function remains an open problem.

References

[1] Agarwal, P. Vázquez, L. and Lenzi, E. K., Recent Trends in Fractional Calculus and Its Applications, Advanced Studies in Complex Systems, Academic Press, London, 2024.

[2] Cahoy, D. O. and Polito, F., Renewal processes based on generalized Mittag–Leffler waiting times, Communications in Nonlinear Science and Numerical Simulation, 18(3) (2013), 639-650.

[3] Carlson, B. C., A connection between elementary and higher transcendental functions, SIAM J. Appl. Math., 17 (1969), 116-148.

[4] Carlson, B. C., Special Functions of Applied Mathematics, Academic Press: New York, NY, USA, 1977.

[5] Connor, R. J. and Mosimann, J. E., Concepts of independence for proportions with a generalization of Dirichlet distribution, J. Am. Stat. Assoc., 64 (1969), 194-206.

[6] deFinetti, B., Theory of Probability, Vol.I, Wiley, New York, 1974.

[7] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Higher Transcendental Functions, Vol. 1, McGraw-Hill Book Corp., New York, 1981.

[8] Gorenflo, R., Kilbas, A. A., Mainardi, F. and Rogosin, S. V., Mittag-Leffler Functions, Related Topics and Applications, Springer Monographs in Mathematics, Springer, Berlin, Heidelberg, 2014.

[9] Hardy, G. H., Littlewood, J. E. and Polya, G., Inequalities, Cambridge University Press, 1952.

[10] Kilbas, A. A., Saigo, M. and Saxena, R. K., Solutions of Volterra integro-differential equations with generalized Mittag-Leffler functions in the kernels, J. Integral Equations Appl., 14(4) (2002), 377-396.

[11] Kilbas, A. A., Saigo, M. and Saxena, R. K., Generalized Mittag-Leffler function and generalized fractional calculus operators, Integral Transforms and Special Functions, 15(1) (2004), 31-49.

[12] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[13] Kilbas, A. A. and Kattuveettill, A., Representations of Dirichlet averages of generalized Mittag-Leffler function via fractional integrals and special functions, Frac. Calc. Appl. Anal., 11 (2008), 471-492.

[14] Kumar, D., Ram, J. and Choi, J., Dirichlet Averages of Generalized Mittag-Leffler Type Function, Fractal Fract., 6 (2022), 297.

[15] Mainardi, F. and Gorenflo, R., Time-Fractional Derivatives in Relaxation Processes: A Tutorial Survey, Fract. Calc. Appl. Anal., 10(3) (2007), 269-308.

[16] Mainardi, F., Fractional Calculus And Waves In Linear Viscoelasticity: An Introduction To Mathematical Models, Imperial College Press, London, 2010.

[17] Mainardi, F., Why the Mittag-Leffler Function Can Be Considered the Queen Function of the Fractional Calculus?, Entropy, 22(12) (2020), 1359.

[18] Mathai, A. M., A Handbook of Generalized Special Functions for Statistical and Physical Sciences, Oxford University Press, Oxford, 1993.

[19] Mathai, A. M., Order statistics from logistic distribution and applications to survival & reliability analysis, IEEE Transactions on Reliability, 52(2) (2003), 200-206.

[20] Ongaro, A. and Migliorati, S., A generalization of the Dirichlet distribution, J. Multivar. Anal., 114 (2013), 412-426.

[21] Prabhakar, T. R., A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 19 (1971), 7-15.

[22] Rogosin, S., The Role of the Mittag-Leffler Function in Fractional Modeling., Mathematics, 3 (2015), 368-381.

[23] Samko S. G., Kilbas, A. A. and Marichev, O. I., Fractional integrals and derivatives: theory and applications, Gordon and Breach Science Publishers, Amsterdam, 1993.

[24] Stamova, I. and Stamov, G., Mittag-Leffler synchronization of fractional neural networks with time-varying delays and reaction–diffusion terms using impulsive and linear controllers, Neural Networks, 96 (2017), 22-32.

[25] Thomas, S. and Jacob, J., A generalized Dirichlet model, Statistics and Probability Letters, 76 (2006), 1761-1767.

[26] Thomas, S. and Mathai, A. M., $p-$Content of a $p-$Parallelotope and Its Connection to Likelihood Ratio Statistic, Sankhyä: The Indian Journal of Statistics, Series A, 71(1) (2009), 49-63.

[27] Wright, E. M., The asymptotic expansion of the generalized hypergeometric function, J. London Math. Soc., 10 (1935), 286-293.

This page intentionally left blank.